Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Probability 1

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Max. points : 50. **Time Limit : 3 hours.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed.
Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points.

1. Suppose we consider a probability space corresponding to equally likely outcomes on a finite set S. Which of the following events are independent ?
   (a) Events A and B such that $A, B \neq \emptyset$ but $A \cap B = \emptyset$.
   (b) Events A,B such that $|A \cap B| = |A||B|$
   (c) Events A, B such that $|A \cup B| = |A||B|$
   (d) Events A, B such that $|A \cap B| = |A| + |B|$
   (e) Events A, B such that $|A \cup B| = |A| + |B|$
   (f) None of the above.

2. Which of the following are true about independent random variables ?
   (a) If $\mathbb{P}(X = c) = 1$ for some real number c, then X is independent of itself.
   (b) If X and Y are independent and Y and Z are independent, then X and Z are independent.
   (c) If X, Y, Z are independent random variables then X and Z are also independent.

(d) If all three pairs (X, Y), (Y, Z) and (X, Z) are independent random variables, then X, Y, Z are also independent.

3. Let there be n distinct routes from Muthurayana Nagar to Indira Nagar. Suppose that each of the routes is blocked with probability p and independently of other routes. Let X_i denote the random variable that route i is not blocked and X be the number of the routes that are not blocked. What are the conditional probabilities $\mathbb{P}[X_1 = 1|X = k]$ and $\mathbb{P}[X \geq 1|X_1 = 1]$ respectively?

   (a) $1, 1$
   (b) $\frac{1}{n}, \frac{1}{k}$
   (c) $\frac{1}{k}, 1$
   (d) $\frac{1}{k}, \frac{1}{n}$
   (e) $\frac{k}{n}, 1$
   (f) $\frac{1}{k}, p$
   (g) $p, 1$
   (h) $p, \frac{1}{k}$
   (i) $\frac{kp}{n}, 1$

4. For $n \geq 5$, let $X_1, \ldots, X_n$ be discrete random variables with all moments being finite. Which of the following hold ?

   (a) $\mathsf{VAR}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \mathsf{VAR}(X_i)$.
   (b) $\mathbb{E}[X_1 + \ldots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i]$.
   (c) $\mathbb{E}[X_1 X_n] \neq \mathbb{E}[X_1]\,\mathbb{E}[X_n]$.
   (d) $\mathbb{E}\left[X_1^k\right] = \mathbb{E}\left[X_n^k\right]$ for all $k \geq 1$ if $X_1, \ldots, X_n$ are identically distributed.
   (e) $\mathbb{E}\left[X_1^k\right] = \mathbb{E}\left[X_n^k\right]$ for all $k \geq 1$ if $X_1, \ldots, X_n$ are independent.
   (f) $\mathbb{E}[X_1 + \ldots + X_n] \geq \sum_{i=1}^{n-3} \mathbb{E}[X_i]$.

5. Which of the following is a CDF ?

   (a) $F(x) = e^{-x}$
   (b) $F(x) = \frac{1}{1+x^2}$
   (c) $F(x) = 1 - \frac{e^{-x}}{2}$
   (d) $F(x) = 1 - 2e^{-|x|}$
   (e) $F(x) = \frac{1}{2} + \frac{x}{2\sqrt{1+x^2}}$
   (f) None of the above.

2 PART B : 40 Points.

Answer any four questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

1. Let $X_1, \ldots, X_n$ be an i.i.d. sequence of discrete random variables and let Y be the minimum of these n random variables.
   (a) Express the CDF of Y in terms of n and F_X, the CDF of X.
   (b) Let $N := \min\{i : X_i = Y\}$. Compute the conditional pmf $p_{N|Y}(.|y)$ and the unconditional pmf $p_N(.)$.
2. Suppose that the number of children N in a family satisfies
$$\mathbb{P}(N = n) = c(\frac{2}{5})^n, \ n = 0, 1, \ldots$$
for some constant $c > 0$. Now suppose that each child in a family is equally likely to be a boy or a girl. Let X be the number of girls in the family. Compute $\mathbb{E}[N], \mathbb{E}[X|N = n]$ and $\mathbb{E}[X]$.
3. Consider the Polya urn model where initially there are r red balls and b black balls respectively where $r, b \in \mathbb{N}$. Let $c \in \mathbb{N}$. At each step, a ball is chosen uniformly at random and c balls of the same colour are added into the urn along with the chosen ball. No ball of the opposite colour is added. Let $X_{l,m,n}$ denote the indicator random variable that a red ball is chosen at the lth, mth and nth step where $l < m < n$. Compute $\mathbb{E}[X_{l,m,n}], \mathsf{VAR}(X_{l,m,n})$ for all $l, m, n \geq 1$.
4. Let $k \geq 1$. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables such that $\mathbb{E}\big[|X_1|^j\big] < \infty$ for all $j \geq 1$. Set $\mu := \mathbb{E}[X_1] \in \mathbb{R}$. For each $i \geq 1$, define $Y_i := \prod_{j=i}^{i+k} X_j = X_i X_{i+1} \ldots X_{i+k}$ and $S_n = \sum_{i=1}^{n} Y_i$. Show that for all $\epsilon > 0$, we have that
$$\lim_{n\to\infty} \mathbb{P}(|\frac{S_n}{n} - \mu^k| \geq \epsilon) = 0.$$
5. *Laplace distribution :* Let $b \in (0, \infty)$ be a parameter. Suppose that X is a random variable with pdf $f(x) := \frac{1}{2b} e^{-\frac{|x|}{b}}$ for $x \in \mathbb{R}$. Show the following :
   (a) Find the pdf of the random variable $|X|$. (**2**)
   (b) Compute all the moments of X. (**5**)
   (c) Find a function g such that $g(U) \stackrel{d}{=} X$ where U is the Uniform $(0, 1)$ random variable. (**3**)